Statistical applications of Large Random Matrix theory to wireless comunication

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joint work with
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Point-to-point wireless communication and MIMO channel

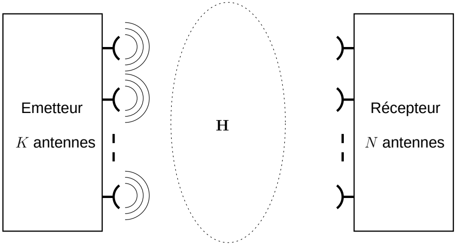

Figure: MIMO channel with K antennas at the transmitter and N antennas at the receiver

The received signal is given by $\mathbf{y} = \mathbf{Hx} + \sigma\mathbf{w}$ where

- H_{ij} is the gain between receiving antenna i and emitting antenna j.

Interference from multiple sources

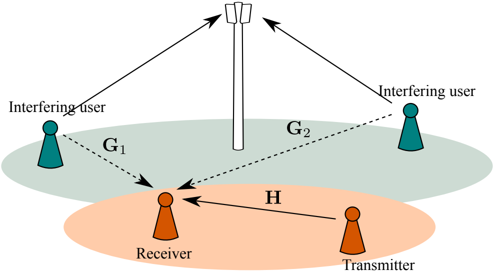

Figure: Users with channels G_1 and G_2 interfere with the communication between the receiver and transmitter

Scenario. In a point-to-point wireless communication, the receiver **undergoes coloured interference** from **multiple sources**, whereas the channel with the transmitter is perfectly known.

Communication model and ergodic capacity

Communication equation.

$$\overline{\mathbf{Y}} = \mathbf{H}\mathbf{X}_0 + \sum_{k=1}^{K} \mathbf{G}_k \mathbf{X}_k + \sigma \mathbf{W}$$

Communication model and ergodic capacity

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$$\overline{\mathbf{Y}} = \mathbf{H}\mathbf{X}_0 + \sum_{k=1}^{K} \mathbf{G}_k \mathbf{X}_k + \sigma \mathbf{W}$$

Observations. During a learning sequence, $\mathbf{X}_0$ is known and $\mathbf{H}$ is estimated, hence the following observations **are available**:

$$\begin{aligned} \mathbf{Y} &= \overline{\mathbf{Y}} - \mathbf{H}\mathbf{X}_0 \\ &= \sum_{k=1}^{K} \mathbf{G}_k \mathbf{X}_k + \sigma \mathbf{W} \quad \stackrel{\triangle}{=} \quad \mathbf{G}\mathbf{X} + \sigma \mathbf{W} \ , \quad \mathbf{G} = [\mathbf{G}_1, \cdots, \mathbf{G}_K]. \end{aligned}$$

Communication model and ergodic capacity

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Associated ergodic capacity.

$$C_{\mathrm{erg}} = \frac{1}{N} \log\det\left(\sigma^2 \mathbf{I} + \mathbf{G}\mathbf{G}^* + \mathbf{H}\mathbf{H}^*\right) - \frac{1}{N} \log\det\left(\sigma^2 \mathbf{I} + \mathbf{G}\mathbf{G}^*\right)$$

Objective

Estimation of the **ergodic capacity**

$$C_{\text{erg}} = \frac{1}{N} \log\det\left(\sigma^2 \mathbf{I} + \mathbf{G}\mathbf{G}^* + \mathbf{H}\mathbf{H}^*\right) - \frac{1}{N} \log\det\left(\sigma^2 \mathbf{I} + \mathbf{G}\mathbf{G}^*\right)$$

based on the $N \times M$ observations

$$\mathbf{Y} = \mathbf{G}\mathbf{X} + \sigma \mathbf{W} .$$

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based on the $N \times M$ observations

$$\mathbf{Y} = \mathbf{G}\mathbf{X} + \sigma \mathbf{W} \ .$$

Regime of interest: M larger but **of the same order** as N:

$$M \propto \rho N \ , \quad \rho > 1.$$

Formally:

$$1 < \liminf \frac{M}{N} \leq \limsup \frac{M}{N} < \infty \ .$$

The traditionnal estimator

Regime where $M >> N$. If $M \to \infty$, N fixed:

$$\frac{1}{M}\mathbb{E}\mathbf{YY}^* = \sigma^2\mathbf{I} + \mathbf{GG}^* \quad \text{and} \quad \frac{1}{M}\mathbf{YY}^* \xrightarrow[M\to\infty, N \text{ fixed}]{a.s.} \sigma^2\mathbf{I} + \mathbf{GG}^*$$

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Hence one expects that:

$$\begin{aligned}\frac{1}{N}\log\det\left(\frac{1}{M}\mathbf{YY}^* + \mathbf{HH}^*\right) - \frac{1}{N}\log\det\left(\sigma^2\mathbf{I} + \mathbf{GG}^* + \mathbf{HH}^*\right) &\to 0\,, \\ \frac{1}{N}\log\det\left(\frac{1}{M}\mathbf{YY}^*\right) - \frac{1}{N}\log\det\left(\sigma^2\mathbf{I} + \mathbf{GG}^*\right) &\to 0\,.\end{aligned}$$

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Hence one expects that:

$$\begin{aligned} \frac{1}{N}\log\det\left(\frac{1}{M}\mathbf{Y}\mathbf{Y}^* + \mathbf{H}\mathbf{H}^*\right) - \frac{1}{N}\log\det\left(\sigma^2\mathbf{I} + \mathbf{G}\mathbf{G}^* + \mathbf{H}\mathbf{H}^*\right) &\to 0\ , \\ \frac{1}{N}\log\det\left(\frac{1}{M}\mathbf{Y}\mathbf{Y}^*\right) - \frac{1}{N}\log\det\left(\sigma^2\mathbf{I} + \mathbf{G}\mathbf{G}^*\right) &\to 0\ . \end{aligned}$$

Definition of $\hat{C}_{\mathrm{trad}}$.

$$\hat{C}_{\mathrm{trad}}(y) = \frac{1}{N}\log\det\left(\frac{1}{M}\mathbf{Y}\mathbf{Y}^* + y\mathbf{H}\mathbf{H}^*\right) - \frac{1}{N}\log\det\left(\frac{1}{M}\mathbf{Y}\mathbf{Y}^*\right)$$

Lemma. If N is fixed and $M \to \infty$, then:

$$\hat{C}_{\mathrm{trad}}(1) - C_{\mathrm{erg}} \to 0.$$

Deterministic equivalents I

Marčenko-Pastur model. If $\mathbf{X}$ in a $N \times M$ matrix with i.i.d. entries

$$\mathbb{E}\mathbf{X}_{ij} = 0, \ \mathrm{var}\mathbf{X}_{ij} = \theta^2$$

We are interested in the limiting behaviour of the spectral measure of $\frac{1}{M}\mathbf{X}\mathbf{X}^*$:

$$L_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_n} \ , \quad (\lambda_n) \text{ eigenvalues } \ \text{of } \frac{1}{M}\mathbf{X}\mathbf{X}^*$$

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Stieltjes transform. It is a convenient transform of the spectral measure L_N and is defined as:

$$\begin{aligned} ST(L_N) &= \frac{1}{N}\sum_{n=1}^{N}\frac{1}{\lambda_n - z} \\ &= \frac{1}{N}\mathrm{trace}\left(-z\mathbf{I} + \frac{1}{M}\mathbf{X}\mathbf{X}^*\right)^{-1} \end{aligned}$$

Deterministic equivalents II

Deterministic equivalent for the Stieltjes transform. The Stieltjes transform of the spectral measure satisfies:

$$\frac{1}{N}\sum_{n=1}^{N}\frac{1}{\lambda_n(\frac{1}{M}\mathbf{X}\mathbf{X}^*)-z}-\mathbf{f}_N(z)\xrightarrow[N,M\to 0]{}0$$

where $\mathbf{f}_N$ satisfies the equation:

$$zc-N\theta^2\mathbf{f}_N^2+\left(z+(c_N-1)\,\theta^2\right)\mathbf{f}_N+1=0,\quad c_N=\frac{N}{M}$$

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Marčenko Pastur distribution.

$$\mathbf{f}_N=ST(\pi_N)$$

with

$$\pi_N(d\lambda)=\left(1-\frac{1}{c_N}\right)^+ + \frac{\sqrt{(\lambda_N^+-\lambda)(\lambda-\lambda_N^-)}}{2c_N\theta^2\lambda}1_{(\lambda_N^-,\lambda_N^+)}d\lambda\ ,\quad c_N=\frac{M}{N}\ .$$

where $\lambda_n^{\pm}=\theta^2(1\pm c_n)^2$.

Deterministic equivalents II

Non-centered model. If $\mathbf{Y} = \frac{1}{\sqrt{N}}\mathbf{X} + \mathbf{A}$. Consider the equation:

$$\delta = \frac{1}{M}\mathrm{trace}\left[-z(1+c_N)\delta + (1-c_N) + \frac{\mathbf{A}\mathbf{A}^*}{1+\delta}\right]^{-1}$$

Then

$$\frac{1}{N}\sum_{n=1}^{N}\frac{1}{\lambda_n(\mathbf{Y}\mathbf{Y}^*) - z} - \delta(z) \xrightarrow[N,M\to 0]{} 0$$

The quantity δ is a deterministic equivalent of the spectral measure

$$L_N = \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_n(\mathbf{Y}\mathbf{Y}^*)} \ .$$

Deterministic equivalents III

Model and quantity of interest.

$$\mathbf{Y} = \mathbf{G}\mathbf{X} + \sigma \mathbf{W} \quad \text{and} \quad \mathbf{Q}(y) = \left(y\,\mathbf{H}\mathbf{H}^* + \frac{1}{M}\mathbf{Y}\mathbf{Y}^* \right)^{-1}$$

Deterministic equivalents III

Model and quantity of interest.

$$\mathbf{Y} = \mathbf{G}\mathbf{X} + \sigma \mathbf{W} \quad \text{and} \quad \mathbf{Q}(y) = \left(y\,\mathbf{H}\mathbf{H}^* + \frac{1}{M}\mathbf{Y}\mathbf{Y}^* \right)^{-1}$$

Fundamental equation. Let $y > 0$. The following equation in $\kappa = \kappa(y)$ admits a unique positive solution:

$$\kappa = \frac{1}{M}\text{trace}\left(\left(\sigma^2 \mathbf{I} + \mathbf{G}\mathbf{G}^* \right) \left(\frac{\sigma^2 \mathbf{I} + \mathbf{G}\mathbf{G}^*}{1+\kappa} + y\mathbf{H}\mathbf{H}^* \right)^{-1} \right)$$

Deterministic equivalents III

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$$\kappa = \frac{1}{M}\mathrm{trace} \left(\left(\sigma^2\mathbf{I} + \mathbf{G}\mathbf{G}^*\right) \left(\frac{\sigma^2\mathbf{I} + \mathbf{G}\mathbf{G}^*}{1+\kappa} + y\mathbf{H}\mathbf{H}^* \right)^{-1} \right)$$

Auxiliary quantity.

$$\mathbf{T}(y) \quad = \quad \left(y\,\mathbf{H}\mathbf{H}^* + \frac{\sigma^2\mathbf{I} + \mathbf{G}\mathbf{G}^*}{1+\kappa} \right)^{-1} ,$$

$\mathbf{T}$ is a deterministic equivalent of $\mathbf{Q}$ as we shall see:

Asymptotic results

Lemma 1. The following convergences hold true:

1. For $y > 0$ and $(\mathbf{U})$ $N \times N$ matrices with uniformly bounded norm:

$$\frac{1}{M}\mathrm{trace}\ \mathbf{UQ}(y) - \frac{1}{M}\mathrm{trace}\ \mathbf{UT}(y) \xrightarrow[N,n\to\infty]{a.s.} 0$$

Asymptotic results

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$$\frac{1}{M}\mathrm{trace}\ \mathbf{UQ}(y) - \frac{1}{M}\mathrm{trace}\ \mathbf{UT}(y) \xrightarrow[N,n\to\infty]{a.s.} 0$$

2. For $y > 0$ and also for $y = 0$:

$$\frac{1}{N}\log\det\left(y\,\mathbf{HH}^* + \frac{1}{M}\mathbf{YY}^*\right) - \frac{1}{N}\log\det\left(y\,\mathbf{HH}^* + \frac{\sigma^2\mathbf{I} + \mathbf{GG}^*}{1+\kappa}\right) - \frac{M}{N}\log(1+\kappa) + \frac{M}{N}\frac{\kappa}{1+\kappa} \to 0$$

Corollary: $\hat{C}_{\mathrm{trad}}$ is not consistent

Lemma 2. Under the regime of interest

$$\hat{C}_{\mathrm{trad}}(y) - \left(\frac{1}{N} \log\det \left(y\mathbf{H}\mathbf{H}^* + \frac{\sigma^2 \mathbf{I} + \mathbf{G}\mathbf{G}^*}{1+\kappa} \right) - \frac{1}{N} \log\det \left(\sigma^2 \mathbf{I} + \mathbf{G}\mathbf{G}^* \right) \right) - \frac{M}{N}\log(1+\kappa) + \frac{M}{N}\frac{\kappa}{1+\kappa} + \frac{N-M}{N}\log\left(\frac{M-N}{M}\right) - 1 \to 0$$

Corollary: $\hat{C}_{\rm trad}$ is not consistent

Lemma 2. Under the regime of interest

$$\hat{C}_{\rm trad}(y) - \left(\frac{1}{N} \log\det\left(y\mathbf{H}\mathbf{H}^* + \frac{\sigma^2 \mathbf{I} + \mathbf{G}\mathbf{G}^*}{1+\kappa} \right) - \frac{1}{N} \log\det\left(\sigma^2 \mathbf{I} + \mathbf{G}\mathbf{G}^* \right) \right) - \frac{M}{N}\log(1+\kappa) + \frac{M}{N}\frac{\kappa}{1+\kappa} + \frac{N-M}{N}\log\left(\frac{M-N}{M}\right) - 1 \to 0$$

Remark: This **substantially** differs from what is expected:

$$\hat{C}_{\rm trad}(1) - C_{\rm erg} = \left(\frac{1}{N} \log\det\left(\frac{1}{M}\mathbf{Y}\mathbf{Y}^* + \mathbf{H}\mathbf{H}^* \right) - \frac{1}{N} \log\det\left(\frac{1}{M}\mathbf{Y}\mathbf{Y}^* \right) \right) - \left(\frac{1}{N} \log\det\left(\sigma^2 \mathbf{I} + \mathbf{G}\mathbf{G}^* + \mathbf{H}\mathbf{H}^* \right) - \frac{1}{N} \log\det\left(\sigma^2 \mathbf{I} + \mathbf{G}\mathbf{G}^* \right) \right) \xrightarrow{\text{NO!}} 0$$

The ergodic capacity

Recall the definition

$$C_{\rm erg} \quad = \quad \frac{1}{N}\log\det\left(\sigma^2\mathbf{I} + \mathbf{G}\mathbf{G}^* + \mathbf{H}\mathbf{H}^*\right) \; - \; \frac{1}{N}\log\det\left(\sigma^2\mathbf{I} + \mathbf{G}\mathbf{G}^*\right)$$

Splitting the ergodic capacity. Write $C_{\rm erg} = C^1_{\rm erg} - C^2_{\rm erg}$ where

$$\begin{aligned} C^1_{\rm erg} &= \frac{1}{N}\log\det\left(\sigma^2\mathbf{I} + \mathbf{G}\mathbf{G}^* + \mathbf{H}\mathbf{H}^*\right) \\ C^2_{\rm erg} &= \frac{1}{N}\log\det\left(\sigma^2\mathbf{I} + \mathbf{G}\mathbf{G}^*\right) \end{aligned}$$

We shall separately estimate the 2 quantities, beginning with $C^2_{\rm erg}$.

Estimation of C^2_{erg}

Applying Lemma 1-2) for $y = 0$:

$$\frac{1}{N}\log\det\left(y\,\mathbf{HH}^* + \frac{1}{M}\mathbf{YY}^*\right) - \frac{1}{N}\log\det\left(y\,\mathbf{HH}^* + \frac{\sigma^2\mathbf{I}+\mathbf{GG}^*}{1+\kappa}\right) - \frac{M}{N}\log(1+\kappa) + \frac{M}{N}\frac{\kappa}{1+\kappa} \to 0$$

yields $\kappa = \frac{N}{M-N}$ and

$$\frac{1}{N}\log\det\left(\sigma^2\mathbf{I}+\mathbf{GG}^*\right) - \frac{1}{N}\log\det\left(\frac{1}{M}\mathbf{YY}^*\right) + \frac{N-M}{N}\log\left(\frac{M-N}{M}\right) - 1 \xrightarrow{a.s.} 0\ .$$

hence the desired result.

Estimation of C^1_{erg}

Recall the definition of C^1_{erg}:

$$C^1_{\mathrm{erg}} = \frac{1}{N} \log\det\left(\sigma^2 \mathbf{I} + \mathbf{G}\mathbf{G}^* + \mathbf{H}\mathbf{H}^*\right)$$

Estimation of C^1_{erg}

Recall the definition of C^1_{erg}:

$$C^1_{\mathrm{erg}} = \frac{1}{N} \log\det\left(\sigma^2 \mathbf{I} + \mathbf{G}\mathbf{G}^* + \mathbf{H}\mathbf{H}^*\right)$$

- A priori, C^1_{erg} **does not only depend** on the eigenvalues of $\mathbf{G}\mathbf{G}^*$, in contrast with C^2_{erg}.
- Hence, it will be difficult to get an estimator simply based on the eigenvalues of the observations $\frac{1}{M}\mathbf{Y}\mathbf{Y}^*$

Outline of the proof

Available result

$$\frac{1}{N}\log\det\left(y\,\mathbf{H}\mathbf{H}^* + \frac{1}{M}\mathbf{Y}\mathbf{Y}^*\right) - \frac{1}{N}\left\{\log\det\left(y\,\mathbf{H}\mathbf{H}^* + \frac{\sigma^2\mathbf{I} + \mathbf{G}\mathbf{G}^*}{1+\kappa}\right) + M\log(1+\kappa) - M\frac{\kappa}{1+\kappa}\right\} \to 0$$

Outline of the proof

Available result

$$\frac{1}{N}\log\det\left(y\,\mathbf{H}\mathbf{H}^* + \frac{1}{M}\mathbf{Y}\mathbf{Y}^*\right) - \frac{1}{N}\left\{\log\det\left(y\,\mathbf{H}\mathbf{H}^* + \frac{\sigma^2\mathbf{I} + \mathbf{G}\mathbf{G}^*}{1+\kappa}\right) + M\log(1+\kappa) - M\frac{\kappa}{1+\kappa}\right\} \to 0$$

1. Link between C^1_{erg} and the observations if $y_\kappa = \frac{1}{1+\kappa}$:

$$C^1_{\mathrm{erg}} - \left(\frac{1}{N}\log\det\left(\frac{1}{M}\mathbf{Y}\mathbf{Y}^* + y_\kappa\mathbf{H}\mathbf{H}^*\right) + \frac{M-N}{N}\log(y_\kappa) + \frac{M}{N}(1-y_\kappa)\right) \to 0 \quad (1)$$

Outline of the proof

Available result

$$\frac{1}{N}\log\det\left(y\,\mathbf{HH}^* + \frac{1}{M}\mathbf{YY}^*\right) - \frac{1}{N}\left\{\log\det\left(y\,\mathbf{HH}^* + \frac{\sigma^2\mathbf{I}+\mathbf{GG}^*}{1+\kappa}\right) + M\log(1+\kappa) - M\frac{\kappa}{1+\kappa}\right\} \to 0$$

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2. Approximation $\hat{y}$ (which depends on the observations!) of y_κ (which depends on the unknown $\mathbf{G}$!)

Outline of the proof

Available result

$$\frac{1}{N}\log\det\left(y\,\mathbf{HH}^* + \frac{1}{M}\mathbf{YY}^*\right) - \frac{1}{N}\left\{\log\det\left(y\,\mathbf{HH}^* + \frac{\sigma^2\mathbf{I}+\mathbf{GG}^*}{1+\kappa}\right) + M\log(1+\kappa) - M\frac{\kappa}{1+\kappa}\right\} \to 0$$

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2. Approximation $\hat{y}$ (which depends on the observations!) of y_κ (which depends on the unknown **G**!)
3. Substitution of y_κ by $\hat{y}$ in (1).

Details on $\hat{y}$ I

Approximation of y_κ. Recall that

$$\kappa = \frac{1}{M}\mathrm{trace}\left((\sigma^2 \mathbf{I} + \mathbf{GG}^*) \left(\frac{\sigma^2 \mathbf{I} + \mathbf{GG}^*}{1+\kappa} + y\mathbf{HH}^* \right)^{-1} \right)$$

Lemma Define $\hat{y}$ by

$$\hat{y} = 1 - \frac{N}{M} + \frac{\hat{y}}{M}\mathrm{trace}\, \mathbf{HH}^* \left(\hat{y}\mathbf{HH}^* + \frac{1}{M}\mathbf{YY}^* \right)^{-1}$$

then $\hat{y} - y_\kappa \to 0$

Details on $\hat{y}$ II

Elements of proof. It is easy to prove that $y = \frac{1}{1+\kappa(y)}$ admits a unique solution y_κ and that

$$
\begin{aligned}
y_\kappa &= 1 - \frac{N}{M} + \frac{1}{M}\mathrm{trace}\ \mathbf{HH}^*\left(\sigma^2\mathbf{I} + \mathbf{GG}^* + \mathbf{HH}^*\right)^{-1} \\
&= 1 - \frac{N}{M} + \frac{y_\kappa}{M}\mathrm{trace}\ \mathbf{HH}^*\left(\frac{\sigma^2\mathbf{I} + \mathbf{GG}^*}{1+\kappa} + y_\kappa\mathbf{HH}^*\right)^{-1} \\
&= 1 - \frac{N}{M} + \frac{y_\kappa}{M}\mathrm{trace}\ \mathbf{HH}^*\mathbf{T}(y_\kappa) \\
&\approx 1 - \frac{N}{M} + \frac{y_\kappa}{M}\mathrm{trace}\ \mathbf{HH}^*\mathbf{Q}(y_\kappa)
\end{aligned}
$$

Hence $\hat{y}$ satisfying

$$
\hat{y} = 1 - \frac{N}{M} + \frac{\hat{y}}{M}\mathrm{trace}\ \mathbf{HH}^*\mathbf{Q}(\hat{y})
$$

is a good candidate to approximate y_κ.

Consistent estimator for C_{erg}

Gathering the 2 estimators for C^1_{erg} and C^2_{erg}, we obtain:

$$C_{\mathrm{erg}} - \hat{C}_G \to 0$$

where

$$\begin{aligned}\hat{C}_G &= \frac{1}{N}\log\det\left(\frac{1}{M}\mathbf{YY}^* + \hat{y}\mathbf{HH}^*\right) - \frac{1}{N}\log\det\left(\frac{1}{M}\mathbf{YY}^*\right) \\ &\quad + \frac{M-N}{N}\left(\log\left(\frac{M\hat{y}}{M-N}\right)+1\right) - \frac{M}{N}\hat{y}\end{aligned}$$

In particular,

$$= \hat{C}_{\mathrm{trad}}(\hat{y}) + \frac{M-N}{N}\left(\log\left(\frac{M\hat{y}}{M-N}\right)+1\right) - \frac{M}{N}\hat{y}$$

A central limit theorem for $\hat{C}_G$

Theorem. Let

$$\Theta_N = 2\log(My_{\kappa}) - \log\left[(M-N)\left(M - \mathrm{trace}\left(\mathbf{I} + \mathbf{HH}^*\left(\mathbf{GG}^* + \sigma^2\mathbf{I}\right)^{-1}\right)^{-2}\right)\right]$$

Then

$$\frac{N}{\Theta_N}\left(\hat{C}_G - C_{\mathrm{erg}}\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1).$$

Elements of proof I

Recall that

$$\hat{C}_G = \hat{C}_{\mathrm{trad}}(\hat{y}) + \frac{M-N}{N}\left(\log\left(\frac{M\hat{y}}{M-N}\right)+1\right) - \frac{M}{N}\hat{y} \tag{2}$$

Elements of proof I

Recall that

$$\hat{C}_G = \hat{C}_{\mathrm{trad}}(\hat{y}) + \frac{M-N}{N}\left(\log\left(\frac{M\hat{y}}{M-N}\right)+1\right) - \frac{M}{N}\hat{y} \tag{2}$$

Estimates for $\hat{y}$. The following estimates hold true:

$$\mathrm{var}\ \hat{y} = \mathcal{O}\left(\frac{1}{N^2}\right) \quad \text{and} \quad \mathbb{E}\hat{y} = y_{\kappa} + \mathcal{O}\left(\frac{1}{N^2}\right)$$

Elements of proof I

Recall that

$$\hat{C}_G = \hat{C}_{\mathrm{trad}}(\hat{y}) + \frac{M-N}{N}\left(\log\left(\frac{M\hat{y}}{M-N}\right)+1\right) - \frac{M}{N}\hat{y} \qquad (2)$$

Estimates for $\hat{y}$. The following estimates hold true:

$$\mathrm{var}\ \hat{y} = \mathcal{O}\left(\frac{1}{N^2}\right) \quad \text{and} \quad \mathbb{E}\hat{y} = y_\kappa + \mathcal{O}\left(\frac{1}{N^2}\right)$$

and enable us to **replace** $\hat{y}$ by y_κ in (2) and

$$\hat{C}_G \approx \hat{C}_{\mathrm{trad}}(y_\kappa) + \frac{M-N}{N}\left(\log\left(\frac{My_\kappa}{M-N}\right)+1\right) - \frac{M}{N}y_\kappa$$

fluctuation-wise.

Elements of proof II

It is therefore sufficient to study the fluctuations of

$$\hat{C}_{\mathrm{trad}}(y_\kappa) = \frac{1}{N}\log\det\left(\frac{1}{M}\mathbf{Y}\mathbf{Y}^* + y_\kappa\mathbf{H}\mathbf{H}^*\right) - \frac{1}{N}\log\det\left(\frac{1}{M}\mathbf{Y}\mathbf{Y}^*\right)$$

Elements of proof II

It is therefore sufficient to study the fluctuations of

$$\hat{C}_{\mathrm{trad}}(y_\kappa) = \frac{1}{N}\log\det\left(\frac{1}{M}\mathbf{YY}^* + y_\kappa \mathbf{HH}^*\right) - \frac{1}{N}\log\det\left(\frac{1}{M}\mathbf{YY}^*\right)$$

This can be performed by following step by step

- W, Hachem, O. Khorunzhiy, P. Loubaton, J. Najim and L. Pastur. *A new approach for capacity analysis of large dimensional multi-antenna channels*. IEEE Inf. Theory, Vol. 54 (9), sept. 2008

where a CLT for

$$\mathcal{I} = \frac{1}{N}\log\det\left(\mathbf{I} + \frac{\mathbf{ZZ}^*}{\rho}\right) \ , \quad \mathbf{Z} = \frac{1}{N}\mathbf{D}^{1/2}\mathbf{X}\tilde{\mathbf{D}}^{1/2}$$

is established.

Conclusion

- By relying on Large Random Matrix theory, in particular on deterministic equivalents associated to particular models, it is possible to build consistent estimates in a case where the number of observations is of the same order as the dimension of each observation.

Conclusion

- By relying on Large Random Matrix theory, in particular on deterministic equivalents associated to particular models, it is possible to build consistent estimates in a case where the number of observations is of the same order as the dimension of each observation.
- The technique presented here can be extended to several other models, **although have to be developed on a case-by-case basis**

A short bibliography on G-estimation/Eigen-inference

- X. Mestre. *Improved Estimation of Eigenvalues and Eigenvectors of Covariance Matrices Using Their Sample Estimates*. IEEE Trans. Inf. Th.; vol 54(11); 2008.
- R. Couillet, M. Debbah, J.W. Silverstein, Z. Bai. *Eigen-inference for enery estimation of multiple sources.* IEEE Trans. Inf. Th.; vol. 57(4); 2011.
- P. Vallet and P. Loubaton. *A G-Estimator for the MIMO channel ergodic capacity.* IEEE International Symposium on Information Theory, 2009.
- A. Kammoun, R. Couillet, J. Najim and M. Debbah. *Performance of capacity inference methods under colored interference.* 2011, submitted - `arXiv:1105.5305`.